Normal mode sound propagation in an ocean with random narrow-band surface waves

G. V. Anand and Mathews K. George
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560 012, India

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Normal mode sound propagation in an isovelocity ocean with random narrow-band surface waves is considered, assuming the root-mean-square wave height to be small compared to the acoustic wavelength. Nonresonant interaction among the normal modes is studied using a straightforward perturbation technique. The more interesting case of resonant interaction is investigated using the method of multiple scales to obtain a pair of stochastic coupled amplitude equations which are solved using the Peano–Baker expansion technique. Equations for the spatial evolution of the first and second moments of the mode amplitudes are also derived and solved. It is shown that, irrespective of the initial conditions, the mean values of the mode amplitudes tend to zero asymptotically with increasing range, the mean-square amplitudes tend towards a state of equipartition of energy, and the total energy of the modes is conserved.

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INTRODUCTION

Acoustic propagation in an ocean with a randomly rough surface has been the subject of several investigations in the past. Kuperman and Ingenito[1] have used a boundary perturbation method based on a small waveheight assumption to determine normal mode attenuation coefficients due to scattering from rough boundaries. McDaniel[2] has derived coupled power equations for calculating the energy transfer between modes due to scattering from the rough ocean floor modeled as a stationary Gaussian process. She has also shown[3] that the coupled mode theory is equivalent to the small wave height theory of Kuperman and Ingenito. Bass *et al.*[4,5] have used the Green's function technique to investigate the influence of boundary perturbation on wave propagation. They have obtained a Dyson-type equation for the average Green's function of a perturbed waveguide and determined the eigenfunctions, phase velocities, average field, and second-order statistical moments of the modes. Kryazhev *et al.*[6] have also invoked the perturbation theory to determine the first two moments of the sound field in an Arctic type surface sound channel with an irregular ice boundary. McDaniel and McCammon[7] have studied the effect of mode coupling due to lateral seabed inhomogeneities on propagation loss and transverse horizontal spatial coherence. Dozier and Tappert[8,9] have derived stochastic coupled amplitude equations and deterministic coupled power equations for the randomly coupled modes in an ocean with random sound-speed fluctuations due to internal waves. They have shown that the modal powers reach a unique equilibrium corresponding to equipartition of energy, irrespective of the initial condition. They have also obtained the statistical distribution of the normal mode amplitudes in the equilibrium regime. Boyles[10] has presented a nonperturbative coupled-mode theory of acoustic propagation in an inhomogeneous ocean with randomly rough surface. Harper and Labianca[11] have proposed the much-used model of a random ocean surface as the sum of sinusoids with random phases, and studied the problem of scattering from such a surface. Kohler and Papanicolaou[12] have provided a discussion of fluctuation phenomena in underwater sound propagation dwelling primarily on perturbation techniques applied to coupled amplitude equations. The works of Papanicolaou and Keller,[13] Papanicolaou,[14] Kohler and Papanicolaou,[15,16] and Brissaud and Frisch[17] provide much insight into the problems of stochastic wave propagation in a waveguide.

The problem of normal mode sound propagation in an isovelocity ocean with sinusoidal surface waves of small amplitude has been investigated by the authors in an earlier paper.[18] In this paper, a more realistic model of the ocean surface is considered by assuming that the surface elevation is a narrow-band Gaussian random process and that the root-mean-square wave height of the surface waves is small compared to the acoustic wavelength. The surface is assumed to be static since the frequencies of surface waves are usually very small compared to that of the acoustic waves. The surface undulations are represented as a quasisinusoidal function whose amplitude and phase are slowly varying random functions of the spatial coordinate. The surface is treated as a perturbation of a plane surface, and the equivalent plane surface boundary conditions are obtained using a Taylor series expansion. The resulting boundary value problem is solved using, as in Ref. 18, the multiple-scale perturbation technique, the perturbation parameter being the ratio of the root-mean-square elevation of the surface wave to the acoustic wavelength. Since the spatial power spectral density of the surface elevation has a narrow bandwidth, the interaction between the surface and acoustic waves becomes resonant if the center wave number of the surface satisfies appropriate phase-matching conditions. Nonresonant interaction leads to small random fluctuations in the amplitude and phase of the acoustic wave. In the case of resonant interaction, two acoustic normal modes get stochastically coupled. The coupled-amplitude equations contain coefficients which are random functions of the independent variable. These equations are solved in series form using the Peano–Baker expansion technique[19] in a manner similar to the decomposition procedure of Adomian.[20] This series is then converted into a specialized exponential of a Gaussian random matrix by the technique of ordering (symmetrizing)[21] the coefficient matrix of the system. After obtaining the solution from the propagator matrix, the moments of the normal mode amplitudes can be expressed in terms of exponentials of the double integrals of the autocorrelation functions of the random coefficients appearing in the coupled amplitude equations. It is shown that the first and second moments of the modal amplitudes vary slowly with range X and attain spatial stationarity asymptotically as $X\to\infty$. The mean amplitudes tend to zero while the mean-square amplitudes tend to nonzero limits as $X\to\infty$. It is also observed that the mean-square modal amplitudes satisfy a conservation law which may be interpreted as the law of conservation of acoustic power.

I. FORMULATION OF THE PROBLEM

For the quasistatic surface condition under consideration, the propagation of a plane monochromatic wave of angular frequency ω through an isovelocity ocean is governed by the Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0, \tag{1}$$

where $\psi e^{i\omega t}$ is the velocity potential, x is the horizontal spatial coordinate in the direction of propagation, z is the vertical spatial coordinate positive downwards, and

$$k=\omega/c=2\pi/\lambda, \tag{2}$$

where c is the speed of sound and λ is the acoustic wavelength. The ocean surface can be considered to be a static corrugated surface since the surface wave frequency is usually very small compared to the acoustic frequency. The elevation of the ocean surface from the mean plane $z=0$ is a sample function of a zero-mean narrow-band Gaussian random process $N_1(x)$ whose autocorrelation function is assumed to be

$$R_{N_1}(u)=E[N_1(x)N_1(x-u)]$$
$$=\eta^2\exp[-(\beta^2k_s^2u^2)/2]\cos k_su, \tag{3}$$

$$\beta\ll 1, \tag{4}$$

where k_s is the center wave number (spatial frequency) of the process $N_1(x)$, β is the fractional bandwidth, and

$$\eta^2=E[N_1^2(x)] \tag{5}$$

is the mean-square value of the surface elevation. We can represent $N_1(x)$ in the form

$$N_1(x)=\eta V_1(x)\cos[k_sx+\phi_1(x)], \tag{6}$$

where $V_1(x)$ and $\phi_1(x)$ are slowly varying random functions of x. It follows from Eqs. (5) and (6) that

$$E[V_1(x)]=0,\quad E[V_1^2(x)]=2. \tag{7}$$

We assume a horizontal bottom surface located at $z=h$. The boundary conditions for ψ may therefore be written as

$$\psi(x,z)=0,\quad \text{at } z=N_1(x), \tag{8}$$

$$\frac{\partial\psi(x,z)}{\partial z}=0,\quad \text{at } z=h. \tag{9}$$

The boundary condition at $z=N_1(x)$ can be transformed into a condition at $z=0$ through a Taylor series expansion. We have

$$\psi[x,N_1(x)]=\psi(x,0)+N_1(x)\frac{\partial\psi(x,0)}{\partial z}+\frac{1}{2}N_1^2(x)\frac{\partial^2\psi(x,0)}{\partial z^2}+\cdots. \tag{10}$$

The series on the right-hand side of Eq. (10) can be approximated by the first two terms provided that the root-mean-square value of the third term is small compared to that of the second, i.e., if

$$\left\{E\left[N_1^4(x)\left(\frac{\partial^2\psi(x,0)}{\partial z^2}\right)^2\right]\right\}^{1/2}\ll\left\{E\left[N_1^2(x)\left(\frac{\partial\psi(x,0)}{\partial z}\right)^2\right]\right\}^{1/2}. \tag{11}$$

Since $(\partial\psi/\partial z)=O(k\psi)$, $(\partial^2\psi/\partial z^2)=O(k^2\psi)$, $E[N_1^2(x)]=\eta^2$, and $E[N_1^4(x)]=3\eta^4$, inequality (11) can be recast in the form

$$\epsilon\ll 1, \tag{12}$$

where

$$\epsilon=\eta k=2\pi\eta/\lambda \tag{13}$$

is the surface roughness parameter. Condition (12) implies that the rms value of the surface elevation should be small compared to the acoustic wavelength. Assuming η to be small enough to satisfy condition (12), the boundary condition (8) may be replaced by the approximate boundary condition

$$\psi(x,z)+\eta V_1(x)\left(\frac{\partial\psi(x,z)}{\partial z}\right)\cos[k_sx+\phi_1(x)]=0,\quad \text{at } z=0. \tag{14}$$

To facilitate the use of the perturbation technique we introduce the dimensionless quantities

$$X=kx,\quad Z=kz,\quad H=kh,\quad U=ku,\quad K=k_s/k, \tag{15}$$

and rewrite Eq. (1) and boundary conditions (14) and (9) as

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial Z^2}+\psi=0, \tag{16}$$

$$\psi+\epsilon V(X)\frac{\partial\psi}{\partial Z}\cos[KX+\phi(X)]=0,\quad \text{at } Z=0, \tag{17}$$

$$\frac{\partial\psi}{\partial Z}=0,\quad \text{at } Z=H, \tag{18}$$

where

$$V(X)=V_1(x), \quad \phi(X)=\phi_1(x). \tag{19}$$

The dimensionless parameter ϵ is the normalized rms height of the surface wave. The functions $V(X)$ and $\phi(X)$ vary very slowly as compared to the trigonometric function $\cos KX$, the ratio of the two rates of variation being of order β. In the new coordinate system, the surface elevation is given by

$$N(X)=\epsilon V(X)\cos[KX+\phi(X)] \\ =\epsilon[V_c(X)\cos KX+V_s(X)\sin KX], \tag{20}$$

and the autocorrelation function of $N(X)$ is

$$R_N(U)=E[N(X)N(X-U)] \\ =\epsilon^2 \exp[-(\beta^2K^2U^2)/2]\cos KU. \tag{21}$$

Invoking the following properties of Gaussian random processes:[22]

$$\begin{aligned} &E[V_c(X)V_c(X-U)]=E[V_s(X)V_s(X-U)], \\ &E[V_c(X)V_s(X-U)]=-E[V_s(X)V_c(X-U)], \end{aligned} \tag{22}$$

and combining with Eqs. (20) and (21), we get

$$E[V_c(X)V_c(X-U)]=E[V_s(X)V_s(X-U)] \\ =\exp(-\tfrac{1}{2}\beta^2K^2U^2), \tag{23a}$$

$$E[V_c(X)V_s(X-U)]=E[V_s(X)V_c(X-U)]=0. \tag{23b}$$

We wish to investigate the interaction between the mth normal mode acoustic wave of wave number k_m and the surface wave, where

$$k_m=\{1-[(m-\tfrac{1}{2})(\pi/H)]^2\}^{1/2}, \quad m=1,2,...,M, \tag{24}$$

M being the largest positive integer for which k_m is real. A resonant interaction of order L ($L=1$, 2,...) occurs if

$$|k_m\pm LK|=k_n+\delta, \quad \delta=\epsilon\sigma, \quad \sigma=O(1), \tag{25}$$

for some positive integers L and n, where δ is the detuning parameter. The cases of nonresonant and resonant interactions are investigated separately in the following sections.

II. NONRESONANT INTERACTION

In the case of nonresonant interaction, the boundary value problem represented by Eqs. (16)–(18) can be solved by a simple perturbation technique. On expanding the function $\psi(X,Z)$ in a perturbation series

$$\psi(X,Z)=\psi_0(X,Z)+\epsilon\psi_1(X,Z)+\cdots, \tag{26}$$

substituting into Eqs. (16)–(18), and grouping together terms with like powers of ϵ, we obtain the following hierarchy of boundary-value problems:

Order ϵ^0:

$$\frac{\partial^2\psi_0}{\partial X^2}+\frac{\partial^2\psi_0}{\partial Z^2}+\psi_0=0, \tag{27}$$

$$\psi_0=0, \quad \text{at } Z=0, \tag{28}$$

$$\frac{\partial\psi_0}{\partial Z}=0, \quad \text{at } Z=H; \tag{29}$$

Order ϵ^1:

$$\frac{\partial^2\psi_1}{\partial X^2}+\frac{\partial^2\psi_1}{\partial Z^2}+\psi_1=0, \tag{30}$$

$$\psi_1=-V(X)\frac{\partial\psi_0}{\partial Z}\cos[KX+\phi(X)], \quad \text{at } Z=0, \tag{31}$$

$$\frac{\partial\psi_1}{\partial Z}=0, \quad \text{at } Z=H, \tag{32}$$

⋮ etc.

For investigating the interaction between the acoustic waves and the surface waves, we consider a single mode solution of Eqs. (27)–(29):

$$\psi_0(X,Z)=\tfrac{1}{2}A_m \exp(ik_mX)\sin\alpha_mZ+\text{c.c.}, \tag{33}$$

$$\alpha_m=(m-\tfrac{1}{2})(\pi/H), \quad m=1,2,...,M. \tag{34}$$

Making use of Eq. (33) in Eq. (31), we obtain

$$\psi_1=-\tfrac{1}{2}\alpha_mA_mV(X)\cos[KX+\phi(X)]\exp(ik_mX) \\ +\text{c.c.}, \quad \text{at } Z=0. \tag{35}$$

Thus, the solution of Eq. (30), subject to the boundary conditions (35) and (32), can be written as

$$\begin{aligned}\psi_1(X,Z)=&-\tfrac{1}{4}\alpha_mA_mV(X)\{B_1(Z) \\ &\times\exp i[(k_m+K)X+\phi(X)]+B_2(Z) \\ &\times\exp i[(k_m-K)X-\phi(X)]\}+\text{c.c.}\end{aligned} \tag{36}$$

The substitution of Eq. (36) into Eqs. (30), (35), and (32) yields

$$\frac{d^2B_j}{dZ^2}+q_j^2B_j=0, \quad j=1,2, \tag{37}$$

$$B_j=1, \quad \text{at } Z=0, \quad \frac{dB_j}{dZ}=0, \quad \text{at } Z=H, \quad j=1,2, \tag{38}$$

where

$$q_1^2=1-(k_m+K)^2=-r_1^2, \tag{39a}$$

$$q_2^2=1-(k_m-K)^2=-r_2^2. \tag{39b}$$

In writing Eqs. (37) we have ignored the terms containing derivatives of $V(X)$ and $\phi(X)$ since $V(X)$ and $\phi(X)$ vary very slowly as compared to $\cos KX$. The solution of Eqs. (37) and (38) can be written as

$$B_j=\frac{\cos q_j(H-Z)}{\cos q_jH}=\frac{\cosh r_j(H-Z)}{\cosh r_jH}, \quad j=1,2. \tag{40}$$

Thus, the expression for $\psi(X,Z)$ correct to the first order in ϵ becomes

$$\psi(X,Z)=A_m \sin(\alpha_m Z)\cos(k_m X)-\tfrac{1}{2}\epsilon\alpha_m A_m V(X) \times\{B_1(Z)\cos[(k_m+K)X+\phi(X)] + B_2(Z)\cos[(k_m-K)X-\phi(X)]\}. \quad (41)$$

Equation (41) indicates that the first-order nonresonant interaction results in the generation of two new acoustic modes with small and equal randomly varying amplitudes $\tfrac{1}{2}\epsilon\alpha_m A_m V(X)$. These modes are not normal modes.

Equation (41) can also be written in the form

$$\psi(X,Z)=a_m(X,Z)\cos[k_m X+\epsilon\theta_m(X,Z)], \quad (42)$$

where

$$a_m(X,Z)=A_m \sin \alpha_m Z-\tfrac{1}{2}\epsilon\alpha_m A_m V(X)[B_1(Z)+B_2(Z)] \times\cos[KX+\phi(X)]+O(\epsilon^2). \quad (43)$$

Equations (42) and (43) can be interpreted to mean that the wavy surface induces small random fluctuations in the amplitude of the acoustic wave. The root-mean square value of these fluctuations is equal to $\tfrac{1}{2}\epsilon\alpha_m A_m[B_1(Z)+B_2(Z)]$.

III. RESONANT INTERACTION

For investigating resonant interactions, which result in the generation of new normal modes, we use a singular perturbation technique, viz., the method of multiple scales.[23] We shall assume that either the small parameters ϵ and β are of the same order of magnitude or β is the smaller of the two parameters. The multiple scales are defined as

$$X_n=\epsilon^n X, \quad n=0,1,2,\dots . \quad (44)$$

For determining solutions correct to the first order in ϵ, it is sufficient to use two scales for X, viz., the short scale X_0, characterizing the wavelengths of the propagating modes, and the long scale (or slow scale) X_1 characterizing the spatial modulations of the amplitudes and phases of the modes. The derivatives with respect to X are transformed to

$$\frac{\partial}{\partial X}=\frac{\partial}{\partial X_0}+\epsilon\frac{\partial}{\partial X_1}, \quad (45)$$

$$\frac{\partial^2}{\partial X^2}=\frac{\partial^2}{\partial X_0^2}+2\epsilon\frac{\partial^2}{\partial X_0\,\partial X_1}+\epsilon^2\frac{\partial^2}{\partial X_1^2}. \quad (46)$$

Accordingly, the asymptotic expansion of the solution is now written as

$$\psi(X,Z)=\psi_0(X_0,X_1,Z)+\epsilon\psi_1(X_0,X_1,Z)+\cdots . \quad (47)$$

Equation (20) is also rewritten as

$$N(X)=\epsilon[V_c(X_1)\cos KX_0+V_s(X_1)\sin KX_0], \quad (48)$$

since the cosine component V_c and the sine component V_s are slowly varying functions of X. Substituting Eqs. (46)–(48) into Eqs. (16)–(18) and grouping together terms with like powers of ϵ, we get the following hierarchy of boundary value problems:

Order ϵ^0:

$$\frac{\partial^2\psi_0}{\partial X_0^2}+\frac{\partial^2\psi_0}{\partial Z^2}+\psi_0=0, \quad (49)$$

$$\psi_0=0, \quad \text{at } Z=0, \quad (50)$$

$$\frac{\partial\psi_0}{\partial Z}=0, \quad \text{at } Z=H; \quad (51)$$

Order ϵ^1:

$$\frac{\partial^2\psi_1}{\partial X_0^2}+\frac{\partial^2\psi_1}{\partial Z^2}+\psi_1=-2\frac{\partial^2\psi_0}{\partial X_0\,\partial X_1}, \quad (52)$$

$$\psi_1=-[V_c(X_1)\cos KX_0 + V_s(X_1)\sin KX_0]\frac{\partial\psi_0}{\partial Z}, \quad \text{at } Z=0, \quad (53)$$

$$\frac{\partial\psi_1}{\partial Z}=0, \quad \text{at } Z=H. \quad (54)$$

We assume the following resonance condition:

$$k_m-K=k_n+\delta, \quad \delta=\epsilon\sigma, \quad \sigma=O(1). \quad (55)$$

Accordingly, we consider the following two-mode solution of Eqs. (49)–(51):

$$\psi_0(X_0,X_1,Z)=\tfrac{1}{2}A_m(X_1)\sin(\alpha_m Z)\exp(ik_m X_0) +\tfrac{1}{2}A_n(X_1)\sin(\alpha_n Z)\exp(ik_n X_0)+\text{c.c.}, \quad (56)$$

where the functions $A_m(X_1)$ and $A_n(X_1)$ are chosen so as to satisfy Eqs. (52)–(54). On substituting Eq. (56) into Eqs. (52) and (53) and using Eq. (55), we get

$$\frac{\partial^2\psi_1}{\partial X_0^2}+\frac{\partial^2\psi_1}{\partial Z^2}+\psi_1=-ik_m\frac{dA_m}{dX_1}\exp(ik_m X_0)\sin(\alpha_m Z) -ik_n\frac{dA_n}{dX_1}\exp(ik_n X_0)\sin(\alpha_n Z)+\text{c.c.}, \quad (57)$$

$$\begin{aligned}\psi_1=-\tfrac{1}{4}\{&\alpha_n A_n(V_c-iV_s)\exp i(k_m X_0-\sigma X_1)\\ &+\alpha_m A_m(V_c+iV_s)\exp i(k_n X_0+\sigma X_1)\\ &+\alpha_m A_m(V_c-iV_s)\exp[i(k_m+K)X_0]\\ &+\alpha_n A_n(V_c+iV_s)\exp[i(k_n-K)X_0]\}+\text{c.c.}, \quad \text{at } Z=0.\end{aligned} \quad (58)$$

Hence, the solution of Eq. (57) must be of the form

$$\begin{aligned}\psi_1(X_0,X_1,Z)=i\{&B_m(X_1,Z)\exp(ik_m X_0)\\ &+B_n(X_1,Z)\exp(ik_n X_0)\\ &+D_1(X_1,Z)\exp[i(k_m+K)X_0]\\ &+D_2(X_1,Z)\exp[i(k_n-K)X_0]\}+\text{c.c.}\end{aligned} \quad (59)$$

Substitution of Eq. (59) into Eqs. (57), (58), and (54) gives

$$\frac{\partial^2 B_m}{\partial Z^2}+\alpha_m^2 B_m=-k_m \frac{dA_m}{dX_1}\sin \alpha_m Z, \quad (60)$$

$$B_m=\tfrac{1}{4}\alpha_n A_n(iV_c+V_s)\exp(-i\sigma X_1), \quad \text{at } Z=0, \quad (61)$$

$$\frac{\partial B_m}{\partial Z}=0, \quad \text{at } Z=H, \quad (62)$$

$$\frac{\partial^2 B_n}{\partial Z^2}+\alpha_n^2 B_n=-k_n \frac{dA_n}{dX_1}\sin \alpha_n Z, \quad (63)$$

$$B_n=\tfrac{1}{4}\alpha_m A_m(iV_c-V_s)\exp(i\sigma X_1), \quad \text{at } Z=0, \quad (64)$$

$$\frac{\partial B_n}{\partial Z}=0, \quad \text{at } Z=H, \quad (65)$$

$$\frac{\partial^2 D_1}{\partial Z^2}+[1-(k_m+K)^2]D_1=0, \quad (66)$$

$$D_1=\tfrac{1}{4}\alpha_m A_m(iV_c+V_s), \quad \text{at } Z=0, \quad (67)$$

$$\frac{\partial D_1}{\partial Z}=0, \quad \text{at } Z=H, \quad (68)$$

$$\frac{\partial^2 D_2}{\partial Z^2}+[1-(k_n-K)^2]D_2=0, \quad (69)$$

$$D_2=\tfrac{1}{4}\alpha_n A_n(iV_c-V_s), \quad \text{at } Z=0, \quad (70)$$

$$\frac{\partial D_2}{\partial Z}=0, \quad \text{at } Z=H. \quad (71)$$

On multiplying Eq. (60) by $\sin \alpha_m Z$ and Eq. (63) by $\sin \alpha_n Z$, integrating with respect to Z from 0 to H, and using the boundary conditions (61), (62), (64), and (65), we get

$$\frac{dA_m}{dX_1}=-\frac{\alpha_m\alpha_n}{2k_m H}A_n(V_s+iV_c)\exp(-i\sigma X_1), \quad (72)$$

$$\frac{dA_n}{dX_1}=\frac{\alpha_m\alpha_n}{2k_n H}A_m(V_s-iV_c)\exp(i\sigma X_1). \quad (73)$$

Equations (72) and (73) are a pair of coupled amplitude equations whose solution determines the nature of variation of the amplitudes A_m and A_n. These equations can be written in matrix form as

$$\mathbf{A}'(X_1)=\mathbf{Q}(X_1)\mathbf{A}(X_1), \quad (74)$$

where

$$\mathbf{A}(X_1)=[A_m(X_1)\ A_n(X_1)]^T, \quad (75)$$

$$\mathbf{Q}(X_1)=\begin{bmatrix} 0 & -f_m D(X_1) \\ f_n D^*(X_1) & 0 \end{bmatrix}, \quad (76)$$

$$f_m=\frac{\alpha_m\alpha_n}{2k_m H}, \quad f_n=\frac{\alpha_m\alpha_n}{2k_n H}, \quad (77)$$

$$D(X_1)=\exp[-i\sigma X_1][V_s(X_1)+iV_c(X_1)], \quad (78)$$

the prime denotes differentiation with respect to the argument, superscript T denotes matrix transpose, and $*$ denotes the complex conjugate operator.

To solve Eq. (74), we employ the Peano–Baker expansion[19] to first obtain the fundamental matrix (propagator matrix) $\mathbf{P}(X_1)$ of Eq. (74). The propagator matrix satisfies the same differential equation as Eq. (74), i.e.,

$$\mathbf{P}'(X_1)=\mathbf{Q}(X_1)\mathbf{P}(X_1), \quad (79)$$

with the initial condition

$$\mathbf{P}(0)=\mathbf{I}, \quad (80)$$

where $\mathbf{I}$ is the 2×2 identity matrix. The solution of Eq. (74) can be expressed in terms of the propagator matrix $\mathbf{P}(X_1)$ as

$$\mathbf{A}(X_1)=\mathbf{P}(X_1)\mathbf{A}(0). \quad (81)$$

To determine $\mathbf{P}(X_1)$, we solve Eq. (79) iteratively by setting

$$\begin{aligned} &\mathbf{P}_0=\mathbf{I}, \\ &\mathbf{P}_j'=\mathbf{Q}(X_1)\mathbf{P}_{j-1}, \quad j=1,2,\ldots . \end{aligned} \quad (82)$$

Therefore, we obtain

$$\begin{aligned} &\mathbf{P}_1=\mathbf{I}+\int_0^{X_1}\mathbf{Q}(X_2)dX_2, \\ &\mathbf{P}_2=\mathbf{I}+\int_0^{X_1}\mathbf{Q}(X_2)\mathbf{P}_1(X_2)dX_2, \\ &\vdots \\ &\text{etc.} \end{aligned} \quad (83)$$

In the limit, the propagator matrix is given by the following infinite series

$$\begin{aligned} \mathbf{P}(X_1)=\mathbf{I}&+\int_0^{X_1}\mathbf{Q}(X_2)dX_2 \\ &+\int_0^{X_1}dX_2\int_0^{X_2}dX_3\,\mathbf{Q}(X_2)\mathbf{Q}(X_3) \\ &+\int_0^{X_1}dX_2\int_0^{X_2}dX_3\int_0^{X_3}dX_4\,\mathbf{Q}(X_2) \\ &\times\mathbf{Q}(X_3)\mathbf{Q}(X_4)+\cdots . \end{aligned} \quad (84)$$

The solution given by Eq. (84) is an infinite series of n-fold iterated integrals ($n=1,2,3,\ldots$). This solution satisfies Eqs. (79) and (80), as can be seen by differentiating Eq. (84) term by term assuming that the series is absolutely and uniformly convergent with probability one. The validity of this convergence assumption is proved in Appendix A.

IV. DETERMINATION OF CLOSED FORM EXPRESSIONS FOR MODAL AMPLITUDES

The series solution given by Eq. (84) cannot be directly used for determining the statistical moments of the modal amplitudes $A_m(X_1)$ and $A_n(X_1)$. Hence, we shall try to obtain a closed-form expression for the propagator matrix $\mathbf{P}(X_1)$. We note that the right-hand side of Eq. (84) contains matrix products which are not commutative, e.g.,

$$\mathbf{Q}(X_2)\mathbf{Q}(X_3) \neq \mathbf{Q}(X_3)\mathbf{Q}(X_2), \quad \text{for } X_2 \neq X_3. \tag{85}$$

Taking the third term in the series on the right-hand side of Eq. (84), we have

$$\int_0^{X_1} dX_2 \int_0^{X_2} dX_3\, \mathbf{Q}(X_2)\mathbf{Q}(X_3) = \frac{1}{2!} \int_0^{X_1} dX_2 \int_0^{X_2} dX_3\, \mathbf{Q}(X_2)\mathbf{Q}(X_3) + \frac{1}{2!} \int_0^{X_1} dX_3 \int_{X_3}^{X_1} dX_2\, \mathbf{Q}(X_2)\mathbf{Q}(X_3). \tag{86}$$

However, we have

$$\int_0^{X_1} dX_3 \int_{X_3}^{X_1} dX_2\, \mathbf{Q}(X_2)\mathbf{Q}(X_3) = \int_0^{X_1} dX_2 \int_{X_2}^{X_1} dX_3\, \mathbf{Q}(X_3)\mathbf{Q}(X_2). \tag{87}$$

by a simple interchange of the dummy variables X_2 and X_3. Therefore, we obtain from Eqs. (86) and (87) the following formula:

$$\int_0^{X_1} dX_2 \int_0^{X_2} dX_3\, \mathbf{Q}(X_2)\mathbf{Q}(X_3) = \frac{1}{2!} \int_0^{X_1} dX_2 \int_0^{X_1} dX_3 [\mathbf{Q}(X_2)\mathbf{Q}(X_3)\chi(X_2,X_3) + \mathbf{Q}(X_3)\mathbf{Q}(X_2)\chi(X_3,X_2)], \tag{88}$$

where $\chi(X_2,X_3,\ldots,X_{n+1})$ is the generalized unit step function defined as

$$\chi(X_2,X_3,\ldots,X_{n+1}) = \begin{cases} 1, & \text{if } X_2 > X_3 > \cdots > X_{n+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{89}$$

The somewhat tortuous procedure leading to Eq. (88) is necessary because the matrix product is not commutative. Using the general definition cited in Eq. (89), we obtain the following equation in an analogous fashion:

$$\int_0^{X_1} dX_2 \int_0^{X_2} dX_3 \int_0^{X_3} dX_4\, \mathbf{Q}(X_2)\mathbf{Q}(X_3)\mathbf{Q}(X_4) = \frac{1}{3!} \int_0^{X_1} dX_2 \int_0^{X_1} dX_3 \int_0^{X_1} dX_4\, \mathbf{\Gamma}(X_2,X_3,X_4), \tag{90}$$

where

$$\begin{aligned} \mathbf{\Gamma}(X_2,X_3,X_4) = {} & \mathbf{Q}(X_2)\mathbf{Q}(X_3)\mathbf{Q}(X_4)\chi(X_2,X_3,X_4) + \mathbf{Q}(X_2)\mathbf{Q}(X_4)\mathbf{Q}(X_3)\chi(X_2,X_4,X_3) \\ & + \mathbf{Q}(X_3)\mathbf{Q}(X_4)\mathbf{Q}(X_2)\chi(X_3,X_4,X_2) + \mathbf{Q}(X_3)\mathbf{Q}(X_2)\mathbf{Q}(X_4)\chi(X_3,X_2,X_4) \\ & + \mathbf{Q}(X_4)\mathbf{Q}(X_2)\mathbf{Q}(X_3)\chi(X_4,X_2,X_3) + \mathbf{Q}(X_4)\mathbf{Q}(X_3)\mathbf{Q}(X_2)\chi(X_4,X_3,X_2). \end{aligned} \tag{91}$$

Expressions similar to those on the right-hand sides of Eqs. (88) and (90) can be derived for each and every term of the series on the right-hand side of Eq. (84). Furthermore, Eqs. (88) and (90) can be rewritten, in a compact form, as given below:

$$\int_0^{X_1} dX_2 \int_0^{X_2} dX_3\, \mathbf{Q}(X_2)\mathbf{Q}(X_3) = S\left[\frac{1}{2!} \int_0^{X_1} dX_2 \int_0^{X_1} dX_3\, \mathbf{Q}(X_2)\mathbf{Q}(X_3)\right], \tag{92}$$

and

$$\int_0^{X_1} dX_2 \int_0^{X_2} dX_3 \int_0^{X_3} dX_4\, \mathbf{Q}(X_2)\mathbf{Q}(X_3)\mathbf{Q}(X_4) = S\left[\frac{1}{3!} \int_0^{X_1} dX_2 \int_0^{X_1} dX_3 \int_0^{X_1} dX_4\, \mathbf{Q}(X_2) \times \mathbf{Q}(X_3)\mathbf{Q}(X_4)\right], \tag{93}$$

where the symbol $S\{\cdots\}$ denotes the operation of symmetrizing the iterated integrals. Thus, the symbol $S\{\cdots\}$ serves to make $\mathbf{Q}(\cdot)$ commute for different arguments. Consequently, the iterated integrals can be written, term by term, as expressions that are very compact. Finally, the series given by the right hand side of Eq. (84) can be recast as

$$\mathbf{P}(X_1) = S\left[\sum_{i=0}^{\infty} \frac{1}{i!} \int_0^{X_1} dX_2 \cdots \int_0^{X_1} dX_{i+1} \times \mathbf{Q}(X_2)\cdots\mathbf{Q}(X_{i+1})\right]. \tag{94}$$

Equation (94) can be expressed as a specialized exponential of a matrix, as given below:

$$\mathbf{P}(X_1) = S\left[\exp \int_0^{X_1} \mathbf{Q}(X_2)\,dX_2\right]. \tag{95}$$

In Eq. (95), if the exponential is first expanded in a series, and the S operator is then applied term by term, what is obtained is just the expression on the right-hand side of Eq. (84).

The solution of the coupled amplitude equations [Eq. (74)] can be formally written by substituting Eq. (95) into Eq. (81). The method of obtaining the first and second moments of the modal amplitudes $A_m(X_1)$ and $A_n(X_1)$ is presented in the following sections.

V. DETERMINATION OF THE MEAN

To obtain the means of the mode amplitudes $A_m(X_1)$ and $A_n(X_1)$, we employ the following result obtained in Appendix B:

$$E\left[S\left[\exp\left(\int_0^{X_1} \mathbf{Q}(X_2)dX_2\right)\right]\right]$$

$$=S\left[\exp\left\{\frac{1}{2}\int_0^{X_1} dX_2 \int_0^{X_1} dX_3\, E[\mathbf{Q}(X_2)\mathbf{Q}(X_3)]\right\}\right], \tag{96}$$

where E is the expectation operator. Combining Eqs. (95) and (96) we obtain

$$E[\mathbf{P}(X_1)]=S\left[\exp\left[\frac{1}{2}\int_0^{X_1} dX_2 \times \int_0^{X_1} dX_3\, E[\mathbf{Q}(X_2)\mathbf{Q}(X_3)]\right]\right]. \tag{97}$$

From Eqs. (76), (78), and (23), we get

$$E[\mathbf{Q}(X_2)\mathbf{Q}(X_3)] =\begin{bmatrix} -f_m f_n h(X_2-X_3) & 0 \\ 0 & -f_m f_n h^*(X_2-X_3)\end{bmatrix}, \tag{98}$$

where

$$h(X_2-X_3)=2\exp[-i\sigma(X_2-X_3)-(X_2-X_3)^2/2\gamma^2] \tag{99}$$

and

$$\gamma=\epsilon/\beta K \tag{100}$$

is the correlation radius of the surface elevation in the slow scale X_1.

On substituting Eq. (98) into Eq. (97), we get

$$E[\mathbf{P}(X_1)]=S\left[\exp\left\{\frac{1}{2}\int_0^{X_1} dX_2\int_0^{X_1} dX_3\begin{bmatrix} -f_m f_n h(X_2-X_3) & 0 \\ 0 & -f_m f_n h^*(X_2-X_3)\end{bmatrix}\right\}\right]. \tag{101}$$

The double integral in Eq. (101) can be reduced to a single integral by using the identity

$$\int_0^{X_1} dX_2 \int_0^{X_1} dX_3\, h(X_2-X_3) = \int_{-X_1}^{X_1} (X_1-|u|)h(u)du. \tag{102}$$

On substituting for $h(u)$ from Eq. (99) we find that the imaginary part of the integral in Eq. (102) vanishes, and thus we get

$$\int_0^{X_1} dX_2 \int_0^{X_1} dX_3\, h(X_2-X_3) =4\int_0^{X_1}(X_1-u)\cos(\sigma u)\exp\left(\frac{-u^2}{2\gamma^2}\right)du. \tag{103}$$

We can therefore rewrite Eq. (101) as

$$E[\mathbf{P}(X_1)]=S\left[\exp\left[\int_0^{X_1}\mathbf{\Lambda}(u)du\right]\right] \tag{104a}$$

$$=S\left[\sum_{i=0}^{\infty}\frac{1}{i!}\int_0^{X_1} du_1\cdots \times\int_0^{X_1} du_i\, \mathbf{\Lambda}(u_1)\cdots\mathbf{\Lambda}(u_i)\right], \tag{104b}$$

where

$$\mathbf{\Lambda}(u_i)=-2f_m f_n(X_1-u_i)\cos(\sigma u_i) \times\exp(-u_i^2/2\gamma^2)\mathbf{I},\quad i=1,2,... \tag{105}$$

and $\mathbf{I}$ is the 2×2 identity matrix. The matrices $\mathbf{\Lambda}(u_i)$ are diagonal matrices and hence their products are commutative, i.e.,

$$\mathbf{\Lambda}(u_i)\mathbf{\Lambda}(u_j)=\mathbf{\Lambda}(u_j)\mathbf{\Lambda}(u_i),\quad \text{for all } u_i \text{ and } u_j. \tag{106}$$

Hence, we have

$$S\left[\int_0^{X_1} du_1\cdots\int_0^{X_1} du_i\,\mathbf{\Lambda}(u_1)\cdots\mathbf{\Lambda}(u_i)\right] = \int_0^{X_1} du_1\cdots\int_0^{X_1} du_i\,\mathbf{\Lambda}(u_1)\cdots\mathbf{\Lambda}(u_i), \tag{107}$$

and Eq. (104a) can therefore be written as

$$E[\mathbf{P}(X_1)]=\exp\left(\int_0^{X_1}\mathbf{\Lambda}(u)du\right) =\exp[-2f_m f_n g(X_1)]\mathbf{I}, \tag{108}$$

where

$$g(X_1)=\int_0^{X_1}(X_1-u)\cos(\sigma u)\exp(-u^2/2\gamma^2)du. \tag{109}$$

Using Eqs. (81) and (108), and assuming that the ocean surface perturbations are independent of the initial conditions $A_m(0)$ and $A_n(0)$, we obtain

$$E[A(X_1)]=E[\mathbf{P}(X_1)]E[\mathbf{A}(0)]$$
$$=\exp\{-2f_m f_n g(X_1)\}E[\mathbf{A}(0)]. \quad (110)$$

VI. DETERMINATION OF THE SECOND MOMENTS

We are interested in computing the following second moments:

$$E[A_m(X_1)A_m^*(X_1)],\quad E[A_m(X_1)A_n^*(X_1)],$$
$$E[A_n(X_1)A_m^*(X_1)],\quad \text{and}\quad E[A_n(X_1)A_n^*(X_1)].$$

From Eqs. (72) and (73), we obtain

$$[A_m(X_1)A_m^*(X_1)]'$$
$$=A_m(X_1)[A_m'(X_1)]^*+A_m'(X_1)A_m^*(X_1)$$
$$=-f_m D^*(X_1)A_m(X_1)A_n^*(X_1)$$
$$-f_m D(X_1)A_n(X_1)A_m^*(X_1). \quad (111)$$

Expressions for $[A_m(X_1)A_n^*(X_1)]'$, $[A_n(X_1)A_m^*(X_1)]'$, and $[A_n(X_1)A_n^*(X_1)]'$ can be derived in a similar fashion. This entire set of four equations can be combined into a single matrix equation, viz.,

$$\mathbf{C}'(X_1)=\mathbf{Q}_2(X_1)\mathbf{C}(X_1), \quad (112)$$

where

$$\mathbf{C}(X_1)=[A_m(X_1)A_m^*(X_1),\quad A_m(X_1)A_n^*(X_1),$$
$$A_n(X_1)A_m^*(X_1),\quad A_n(X_1)A_n^*(X_1)]^T, \quad (113)$$

and

$$\mathbf{Q}_2(X_1)=\begin{bmatrix} 0 & -f_m D^*(X_1) & -f_m D(X_1) & 0 \\ f_n D(X_1) & 0 & 0 & -f_m D(X_1) \\ f_n D^*(X_1) & 0 & 0 & -f_m D^*(X_1) \\ 0 & f_n D^*(X_1) & f_n D(X_1) & 0 \end{bmatrix}. \quad (114)$$

Since Eq. (112) has the same form as Eq. (74), the solution of Eq. (112) can be written as

$$\mathbf{C}(X_1)=\mathbf{P}_2(X_1)\mathbf{C}(0), \quad (115)$$

where $\mathbf{P}_2(X_1)$ is the propagator matrix given by

$$\mathbf{P}_2(X_1)=S\left[\exp\int_0^{X_1}\mathbf{Q}_2(X_2)dX_2\right]. \quad (116)$$

Following a procedure analogous to that used to derive Eq. (104a), we can obtain the following expression for $E[\mathbf{P}_2(X_1)]$:

$$E[\mathbf{P}_2(X_1)]=S\left[\exp\left(\int_0^{X_1}\Lambda_2(u)du\right)\right], \quad (117)$$

where

$$\Lambda_2(u)=-2f_m f_n(X_1-u)\cos\sigma u\exp(-u^2/2\gamma^2)$$
$$\times\begin{bmatrix} 1 & 0 & 0 & -f_m/f_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -f_n/f_m & 0 & 0 & 1 \end{bmatrix}. \quad (118)$$

It can be easily verified that the matrix product $\Lambda_2(u_i)\Lambda_2(u_j)$ is commutative for $u_i \neq u_j$. Hence, Eq. (117) can be replaced by

$$E[\mathbf{P}_2(X_1)]=\exp\left(\int_0^{X_1}\Lambda_2(u)du\right). \quad (119)$$

Using the expansion

$$\exp(\mathbf{M})=\mathbf{I}+\mathbf{M}+\frac{\mathbf{M}^2}{2!}+\frac{\mathbf{M}^3}{3!}+\cdots, \quad (120)$$

Eq. (119) can be reduced to

$$E[\mathbf{P}_2(X_1)] = \begin{bmatrix} \frac{1}{2}\{\exp[2q(X_1)]+1\} & 0 & 0 & -(f_m/2f_n)\{\exp[2q(X_1)]-1\} \\ 0 & \exp[q(X_1)] & 0 & 0 \\ 0 & 0 & \exp[q(X_1)] & 0 \\ -(f_n/2f_m)\{\exp[2q(X_1)]-1\} & 0 & 0 & \frac{1}{2}\{\exp[2q(X_1)]+1\} \end{bmatrix}, \tag{121}$$

where

$$q(X_1) = -2f_m f_n g(X_1). \tag{122}$$

Assuming once again that the initial conditions are statistically independent of the surface perturbations, we can write from Eq. (115)

$$E[\mathbf{C}(X_1)] = E[\mathbf{P}_2(X_1)]E[\mathbf{C}(0)]. \tag{123}$$

Combining Eqs. (113), (121), and (123), we can therefore write

$$\begin{aligned} E[|A_m(X_1)|^2] &= \tfrac{1}{2}\{1+\exp[-4f_m f_n g(X_1)]\}E[|A_m(0)|^2] \\ &\quad + \tfrac{1}{2}(f_m/f_n)\{1-\exp[-4f_m f_n g(X_1)]\} \\ &\quad \times E[|A_n(0)|^2], \end{aligned} \tag{124}$$

$$E[A_m(X_1)A_n^*(X_1)] = \exp[-2f_m f_n g(X_1)]E[A_m(0)A_n^*(0)]. \tag{125}$$

Expression for $E[|A_n(X_1)|^2]$ can be obtained by interchanging m and n in Eq. (124).

We have by definition [see Eq. (77)]

$$f_m/f_n = k_n/k_m. \tag{126}$$

Substituting Eq. (126) into Eq. (124), we find that the second moments of the normal mode amplitudes satisfy the conservation law

$$\begin{aligned} &k_m E[|A_m(X_1)|^2] + k_n E[|A_n(X_1)|^2] \\ &\quad = k_m E[|A_m(0)|^2] + k_n E[|A_n(0)|^2]. \end{aligned} \tag{127}$$

The quantity $k_m E[|A_m(X_1)|^2]$ is proportional to the power in the mth mode, and hence Eq. (127) represents the law of conservation of power.

VII. RESULTS AND DISCUSSION

The evolution of the first and second moments of the modal amplitudes under conditions of resonant coupling is governed by Eqs. (110), (124), and (125). Since the evolution of the moments is determined primarily by the behavior of the function $g(X_1)$ defined by Eq. (109), it is instructive to study the properties of this function in some detail. We shall henceforth denote this function by $g(X_1;\sigma,\gamma)$ to make its dependence on the parameters σ and γ explicit. It can be readily shown that

$$\begin{aligned} g(X_1;0,\gamma) &= (\pi/2)^{1/2}\gamma X_1 \operatorname{erf}(X_1/\sqrt{2}\gamma) - \gamma^2 \\ &\quad \times [1-\exp(-X_1^2/2\gamma^2)], \end{aligned} \tag{128}$$

where

$$\operatorname{erf}(v) = (2/\sqrt{\pi}) \int_0^v \exp(-u^2)\,du \tag{129}$$

is the error function. The right-hand side of Eq. (128) is a monotonically increasing function of $|X_1|$ for $|X_1| > \sqrt{2}\gamma$. It can also be shown that, when $\sigma \neq 0$,

$$g(X_1;\sigma,\gamma) = \sigma^{-2}G(\sigma X_1;\sigma\gamma), \quad \sigma \neq 0, \tag{130}$$

where

$$G(y;\alpha) = \int_0^y (y-u)\cos u \exp\left(\frac{-u^2}{2\alpha^2}\right)du. \tag{131}$$

It is evident from Eq. (130) that the nature of variation of $g(X_1;\sigma,\gamma)$ depends on the product $\sigma\gamma$ and the parameter σ acts only as a coordinate stretching factor. Hence, defining

$$y = \sigma X_1, \quad \alpha = \sigma\gamma, \tag{132}$$

the behavior of the two-parameter family of curves $g(X_1;\sigma,\gamma)$ can be deduced from that of the one-parameter family $G(y;\alpha)$.

It can be easily seen from Eq. (131) that $G(y;\alpha)$ is an even function of both y and α. Variation of $G(y;\alpha)$ with y is shown in Fig. 1 for different values of the parameter α. For the limiting values of $\alpha=0$ and $|\alpha/y| \to \infty$, we can derive the results

$$G(y;0) = 0, \quad \text{for all } y, \tag{133}$$

$$\lim_{|\alpha/y| \to \infty} G(y;\alpha) = 2\sin^2(\tfrac{1}{2}y). \tag{134}$$

Evidently, we have

$$G(y;\alpha) = 0, \quad \text{at } y=0, \quad \text{for all } \alpha. \tag{135}$$

The asymptotic behavior of $G(y;\alpha)$ for large $|y|$ is given by

$$\begin{aligned} G(y;\alpha) &\sim (\pi/2)^{1/2}\exp(-\tfrac{1}{2}\alpha^2)|\alpha y|, \\ &\quad \text{as } y \to \infty, \quad |\alpha/y| < \infty. \end{aligned} \tag{136}$$

In accordance with Eq. (135), each curve in Fig. 1 tends to become a straight line with slope $(\pi/2)^{1/2}|\alpha|\exp(-\frac{1}{2}\alpha^2)$, as $y \to \infty$. This limiting slope has the maximum value of $(\pi/2e)^{1/2}$ when $|\alpha|=1$. Hence, the growth of $G(y;\alpha)$ is the fastest when $|\alpha|=1$. For $|\alpha|>5$, the value of this limiting slope is so small that $G(y;\alpha)$ appears to re-

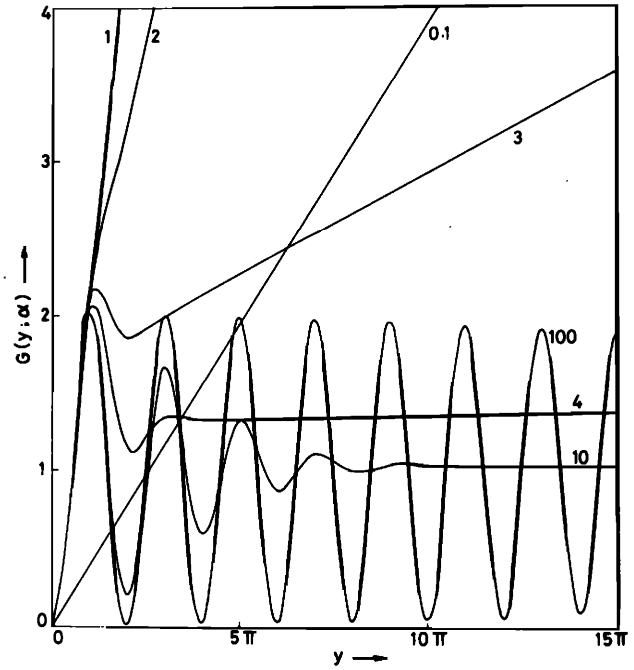

FIG. 1. Plot of the function $G(y;\alpha)$ for different values of α.

main almost constant beyond a certain value of y. The growth of $G(y;\alpha)$ is monotonic if $|\alpha| < (2\pi)^{1/2}$. When $|\alpha| > (2\pi)^{1/2}$, the initial portion of the curve $G(y;\alpha)$ looks like a damped sinusoid oscillating with period 2π and unit initial amplitude. The oscillations occur about a mean value which keeps increasing slowly starting from an initial value of unity. Eventually, the oscillations die out and $G(y;\alpha)$ keeps growing monotonically. The decay of the oscillations becomes progressively slower with increasing $|\alpha|$, and as $|\alpha| \to \infty$, $G(y;\alpha)$ approaches the undamped sinusoid defined by Eq. (134).

Using Eqs. (130) and (126), expressions for $E[A_m]$, $E[|A_m|^2]$, and $E[A_m A_n^*]$ given by Eqs. (110), (124), and (125) can be rewritten as

$$E[A_m(X)] = \exp[-2f_m f_n \sigma^{-2} G(\delta X;\alpha)] E[A_m(0)], \tag{137}$$

$$\begin{aligned} E[|A_m(X)|^2] &= \tfrac{1}{2}(1 + \{E[A_m(X)]/E[A_m(0)]\}^2) E[|A_m(0)|^2] \\ &\quad + (f_m/2f_n)(1 - \{E[A_m(X)]/E[A_m(0)]\}^2) \\ &\quad \times E[|A_n(0)|^2], \end{aligned} \tag{138}$$

$$\begin{aligned} E[A_m(X)A_n^*(X)] &= \{E[A_m(X)]/E[A_m(0)]\} E[A_m(0)A_n^*(0)], \end{aligned} \tag{139}$$

where $X = kx$ is the normalized distance and $\delta = \epsilon\sigma$ is the detuning parameter. Since $G(\delta X;\alpha) \to \infty$ as $X \to \infty$, it follows from Eqs. (137)–(139) that the mean amplitudes $E[A_m]$ and the correlations $E[A_m A_n^*]$ decay to zero and the mean square amplitude $E[|A_m|^2]$ approaches the limiting value $\frac{1}{2}\{E[|A_m(0)|^2] + (f_m/2f_n)E[|A_n(0)|^2]\}$ as $X \to \infty$. The progressive decay of $E[A_m]$ while $E[|A_m|^2]$ remains nonzero signifies that A_m tends to become more random with increasing X. The mean-square amplitudes $E[|A_m(X)|^2]$ at large X may be either larger or smaller than the corresponding initial values. The conservation law given by Eq. (127) is satisfied for all X.

Numerical results for the variation of the mean and mean-square values of the modal amplitudes have been computed for an isovelocity ocean with rigid bottom with the following parameters: sound speed $c = 1500$ m s^{-1}, mean ocean depth $h = 30$ m, frequency $f = 50$ Hz, acoustic wavelength $\lambda = 30$ m, $k = 2\pi f/c = (\pi/15)$ m^{-1}. At this frequency only two normal modes exist with $k_1 = \sqrt{15}/4$ and $k_2 = \sqrt{7}/4$. It is assumed that the initial mode amplitudes are $A_1(0) = 1$, $A_2(0) = 0$. It follows from Eqs. (137) and (138) that $E[A_2] = 0$ for all X, and that $E[A_1] \to 0$, $E[|A_1|^2] \to \frac{1}{2}$ and $E[|A_2|^2] \to \frac{1}{2}(k_1/k_2)$ as $X \to \infty$. The decay of $E[A_1]$ and $E[|A_1|^2]$ from their initial values of unity, and the growth of $E[|A_2|^2]$ from its initial value of zero may be either monotonic or oscillatory, depending on the value of α. The evolution of these averages is controlled by three parameters, viz., α, δ, and σ. These parameters are, in turn, related to the surface wave parameters η (root-mean-square wave height), λ_s (wavelength) and β (fractional bandwidth) as follows:

$$\alpha = (\lambda_s/\beta)(\lambda_{12}^{-1} - \lambda_s^{-1}), \tag{140}$$

$$\delta = \lambda(\lambda_{12}^{-1} - \lambda_s^{-1}), \tag{141}$$

$$\sigma = (\lambda^2/2\pi\eta)(\lambda_{12}^{-1} - \lambda_s^{-1}), \tag{142}$$

where

$$\lambda_{12} = \lambda(k_1 - k_2)^{-1} \tag{143}$$

is the resonance wavelength. When $\lambda_s = \lambda_{12}$, the detuning parameter $\delta = 0$ and coupling between the modes is the strongest. The variation of the moments with X is shown in Figs. 2 and 3 for $\eta = 0.6501$ m, $\lambda_s = 100$ m, and different values of β, viz., $\beta = 0.2269$, 0.02269, 0.002269, and 0.0002269. The corresponding values of σ, δ, and α are $\sigma = 0.05$, $\delta = 0.006808$, and $\alpha = 0.1$, 1, 10, and 100. The decay of $E[A_1]$ and $E[|A_1|^2]$ and the growth of $E[|A_2|^2]$ are monotonic if $|\alpha| < (2\pi)^{1/2}$, i.e., if β exceeds a critical value β_c given by

$$\beta_c = (2\pi)^{-1/2}(\lambda_{12}^{-1}\lambda_s - 1). \tag{144}$$

The monotonic decay/growth are fastest for $\alpha = 1$, i.e., for $\beta = (2\pi)^{1/2}\beta_c$. If $\beta < \beta_c$, the initial variation is oscillatory in nature. The oscillatory variation persists for longer and longer distances as β becomes smaller and smaller. In the limit, as $\beta \to 0$, the variation approaches the periodic behavior characteristic of a sinusoidal surface.[18]

If the root-mean-square wave height η is varied keeping λ_s and β fixed, we find from Eqs. (140)–(142) that σ varies in inverse proportion to η while α, δ remain unchanged. Referring to Eq. (137), it follows that the rate of change of $E[A_1]$ with X becomes faster as η is increased. This effect is demonstrated by the curves in Figs. 4 and 5 which show the variation of $E[A_1]$ with X for $\lambda_s = 100$ m, $\beta = 0.02269$, 0.002269, and $\eta = 0.1$, 0.3, and 1 m. These

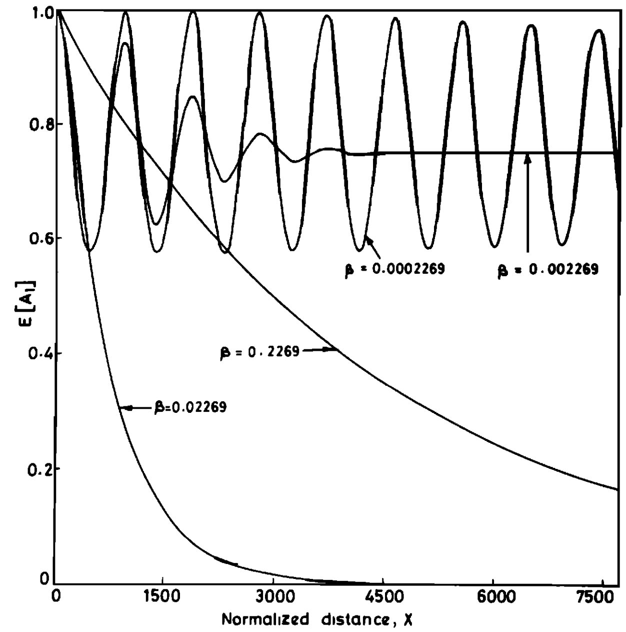

FIG. 2. Variation of $E[A_1]$ with X for $\eta=0.6501$ m, $\lambda_s=100$ m, and different values of β.

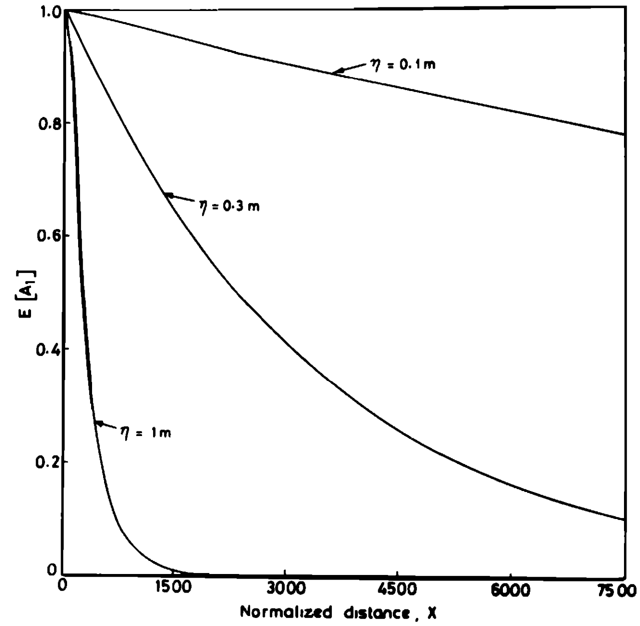

FIG. 4. Variation of $E[A_1]$ with X for $\lambda_s=100$ m, $\beta=0.02269$, and different values of η.

values correspond to $\delta=0.006808$, $\alpha=1$, 10, and $\sigma=0.3251$, 0.1084, and 0.03251, respectively. Using Eq. (138) in conjunction with Figs. 4 and 5, the variation of $E[|A_1|^2]$ and $E[|A_2|^2]$ with X can be readily deduced. It may be noted that the rate of change of the second moments also becomes faster as η is increased.

The effect of variation of surface wavelength λ_s keeping η and β fixed is shown in Fig. 6. When $\lambda_s=\lambda_{12}$, there is perfect resonance and the parameters α, δ, and σ are all equal to zero. As $|\lambda_s-\lambda_{12}|$ is increased, the magnitudes of α, δ, and σ increase monotonically. For the chosen values of λ, k_1, and k_2, we have $\lambda_{12}=97.7810$ m. The curves in Fig. 6 show the variation of $E[A_1]$ with X for $\eta=0.5$ m, $\beta=0.02269$ and $\lambda_s=100$, 102, 105, and 110 m. The corresponding values of α, δ, and σ are as follows: $\alpha=1.0000$, 1.9016, 3.2538, and 5.5074, $\delta=0.006808$, 0.01269, 0.02109, and 0.03408, and $\sigma=0.05000$, 0.09320, 0.1549, and 0.2503. When λ_s is close to λ_{12}, the mean and mean-square amplitudes vary rapidly and monotonically. As λ_s moves away from resonance, the variation becomes slower and also acquires an oscillatory character. Qualitatively similar results are obtained for $\lambda_s<\lambda_{12}$.

VIII. CONCLUSIONS

In this paper, the propagation of plane normal mode acoustic waves in an isovelocity ocean with narrow-band

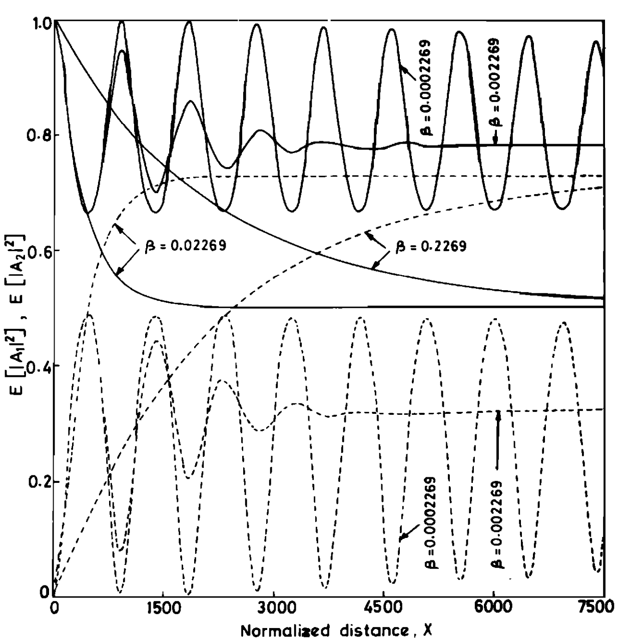

FIG. 3. Variation of $E[|A_1|^2]$ and $E[|A_2|^2]$ with X for $\eta=0.6501$ m, $\lambda_s=100$ m, and different values of β: ——— $E[|A_1|^2]$, ------ $E[|A_2|^2]$.

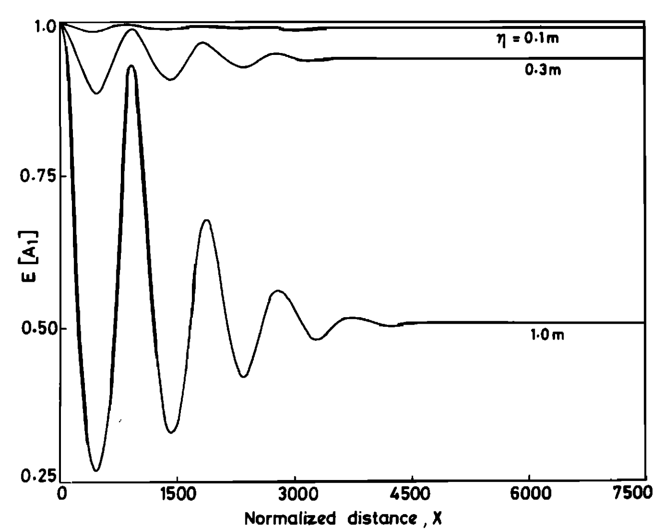

FIG. 5. Variation of $E[A_1]$ with X for $\lambda_s=100$ m, $\beta=0.002269$, and different values of η.

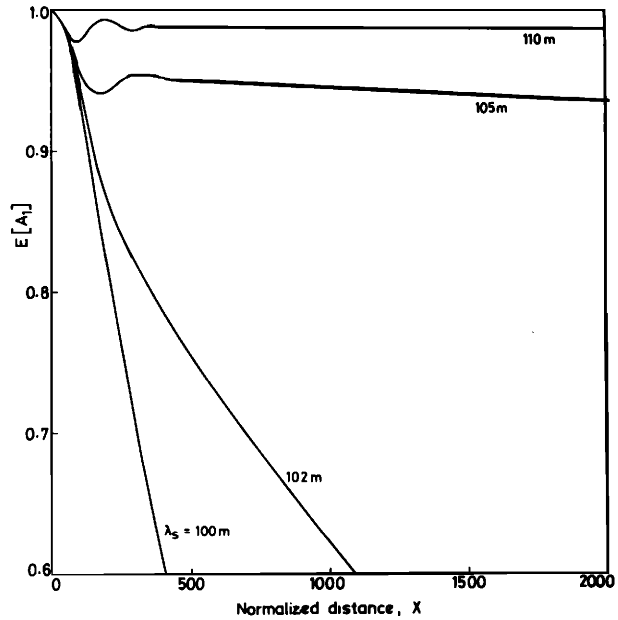

FIG. 6. Variation of $E[A_1]$ with X for $\eta=0.5$ m, $\beta=0.02269$, and different values of λ_s.

random Gaussian surface undulations has been investigated using a perturbation technique under the assumption that the root-mean-square wave height of the surface waves is small compared to the acoustic wavelength. If a normal mode is not resonantly coupled to any other mode, the amplitude of the uncoupled mode suffers small random fluctuations as it propagates in the perturbed channel. In this case, the moments of the modal amplitudes do not vary with propagation distance; the mean amplitude is equal to the amplitude in the unperturbed channel while the standard deviation of the amplitude is of order ϵ. But if two modes are resonantly coupled, the amplitudes of the coupled modes may suffer large random fluctuations. Moreover, in this case, the first and second moments of the modal amplitudes vary with propagation distance X. This variation has several interesting features. The mode amplitudes approach wide-sense stationarity asymptotically; specifically, the mean amplitudes decay and tend to zero as $X\rightarrow\infty$ while the mean-square amplitudes approach nonzero limiting values. If the fractional bandwidth β is sufficiently large, the moments of the modal amplitudes approach their limiting values monotonically. But if β is less than a critical value β_c, the initial variation of the moments is oscillatory. This oscillatory variation is sustained over larger and larger distances as β is reduced, and as $\beta\rightarrow 0$, the variation approaches the periodic behavior characteristic of a channel with a sinusoidal surface.[1] The rate of change of the moments with X increases as the root-mean-square wave height η is increased. The rate of change of the moments is large if the surface wavelength λ_s is close to the resonance wavelength, and reduces rapidly as λ_s moves away from resonance. The mean square amplitudes of the interacting modes satisfy a conservation law which is equivalent to the law of conservation of acoustic power.

In this paper we have assumed rigid boundary conditions at the ocean bottom as a matter of convenience. But a similar analysis, with trivial modifications, can be used for other bottom boundary conditions as well.

A more interesting problem would be the one involving resonant coupling among three or more modes in a multimode channel. In principle, this is a straight forward generalization of the present problem, but the algebraic difficulties may increase rapidly with increase in the system order.

In general, the method developed in this paper is a novel scheme for analytically deriving the moment equations from a linear system of stochastic differential equations with parametric excitation. This method is suitable for solving a class of stochastic boundary value problems with random medium/boundary perturbations.

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APPENDIX A: CONVERGENCE OF THE SERIES IN EQ. (84)

We define the modulus of a matrix $\mathbf{A}$ as the matrix whose elements are the moduli of the corresponding elements of $\mathbf{A}$, i.e.,

$$\text{mod}\,\mathbf{A}=\begin{bmatrix}|A_{11}| & \cdots & |A_{1n}|\\ \vdots & & \vdots\\ |A_{m1}| & \cdots & |A_{mn}|\end{bmatrix}\tag{A1}$$

Given two real $m\times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A}\leqslant\mathbf{B}$ if $A_{ij}\leqslant B_{ij}$ for all i and j. Applying these definitions to the matrix $\mathbf{Q}(X_1)$, we can write

$$\text{mod}\,\mathbf{Q}(X_1)\leqslant\mathbf{H}T(X_1),\quad\text{for all } X_1,\tag{A2}$$

where $T(X_1)$ is the largest element of mod $\mathbf{Q}(X_1)$, i.e.,

$$T(X_1)=\max_{i,j}[\,|Q_{ij}(X_1)|\,],\tag{A3}$$

and $\mathbf{H}$ is the 2×2 matrix with all elements equal to unity, i.e.,

$$\mathbf{H}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.\tag{A4}$$

It can be easily seen that

$$\mathbf{H}^2=2\mathbf{H}.\tag{A5}$$

Combining Eq. (A2) with the first member of Eqs. (83), we get

$$\text{mod}\,\mathbf{P}_1=\text{mod}\left[\mathbf{I}+\int_0^{X_1}\mathbf{Q}(s)ds\right]\leqslant\mathbf{I}+\int_0^{X_1}[\text{mod}\,\mathbf{Q}(s)]ds\leqslant\mathbf{I}+\mathbf{H}U(X_1),\tag{A6}$$

where

$$U(X_1)=\int_0^{X_1} T(s)ds. \tag{A7}$$

For all $X_1 \geqslant 0$, the function $U(X_1)$ is a non-negative real function that increases monotonically from $U(0)=0$.

Next, we prove the following:

Lemma: The inequality

$$\text{mod } \mathbf{P}_n(X_1) \leqslant \mathbf{I}+\mathbf{H} \sum_{i=1}^{n} \frac{2^{i-1}U^i(X_1)}{i!} \tag{A8}$$

is true for all integers $n \geqslant r$ if it is true for $n=r$.

Proof: Considering the rth member of Eqs. (83) in conjunction with Eq. (A2) we can write

$$\begin{aligned}\text{mod } \mathbf{P}_{r+1}(X_1) &= \text{mod}\left(\mathbf{I}+\int_0^{X_1} \mathbf{Q}(s)\mathbf{P}_r(s)ds\right) \\ &\leqslant \mathbf{I}+\int_0^{X_1} [\text{mod } \mathbf{Q}(s)][\text{mod } \mathbf{P}_r(s)]ds \\ &\leqslant \mathbf{I}+\int_0^{X_1} \mathbf{H}T(s)\left(\mathbf{I}+\mathbf{H}\sum_{i=1}^{r} \frac{2^{i-1}U^i(s)}{i!}\right)ds\end{aligned} \tag{A9}$$

if inequality (A8) is true for $n=r$. On integrating the product $U^i(s)T(s)$ by parts and invoking Eq. (A7), we get

$$\int_0^{X_1} U^i(s)T(s)ds=\frac{U^{i+1}(X_1)}{i+1}. \tag{A10}$$

On substituting Eq. (A10) into Eq. (A9) and invoking Eqs. (A5) and (A7), we get

$$\begin{aligned}\text{mod } \mathbf{P}_{r+1}(X_1) &\leqslant \mathbf{I}+\mathbf{H}\left(U(X_1)+\sum_{i=1}^{r} \frac{2^i U^{i+1}(X_1)}{(i+1)!}\right) \\ &= \mathbf{I}+\mathbf{H}\sum_{i=1}^{r+1} \frac{2^{i-1}U^i(X_1)}{i!}.\end{aligned} \tag{A11}$$

It follows by induction that inequality (A8) is true for all integers $n \geqslant r$ if is true for $n=r$.

But inequality (A6) indicates that (A8) is true for $n=1$. Hence inequality (A8) is true for all positive integers n. In the limit, as $n \to \infty$, inequality (A8) yields

$$\lim_{n\to\infty} \text{mod}\mathbf{P}_n(X_1) \leqslant \mathbf{I}+\tfrac{1}{2}\mathbf{H}(e^{2U(X_1)}-1). \tag{A12}$$

Inequality (A12) implies that the sequence $\mathbf{P}_n(X_1)$, which is the sequence of partial sums of the series in Eq. (84), is absolutely bounded with probability 1. Hence, the series in Eq. (84) is absolutely and uniformly convergent with probability 1.

APPENDIX B: PROOF OF EQ. (96)

The characteristic functional of a random process $Y(x)$ is defined as

$$G([k])=E\left[\exp\left(i\int_{-\infty}^{\infty} k(x)Y(x)dx\right)\right], \tag{B1}$$

where $k(x)$ is an arbitrary auxiliary test function. The notation $G([k])$ emphasizes the fact that G depends on the whole function k, not just on the value it takes at any particular point x. The characteristic functional can also be expressed as[24]

$$\log G([k])=\sum_{n=1}^{\infty}\left(\frac{i^n}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} k(x_1)\cdots k(x_n) \times C[Y(x_1),\ldots,Y(x_n)]dx_1\cdots dx_n\right)$$

or

$$G([k])=\exp\left[\sum_{n=1}^{\infty}\left(\frac{i^n}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} k(x_1)\cdots k(x_n) \times C[Y(x_1),\ldots,Y(x_n)]dx_1\cdots dx_n\right)\right], \tag{B2}$$

where $C[Y(x_1),\ldots,Y(x_n)]$ are the cumulants of the joint distribution of $Y(x_1),\ldots,Y(x_n)$ for $n=1, 2,\ldots$. Choosing

$$k(x)=\begin{cases}-i, & 0<x<X_1,\\ 0, & \text{elsewhere},\end{cases} \tag{B3}$$

and equating the right-hand sides of Eqs. (B1) and (B2), we get

$$E\left[\exp\left(\int_0^{X_1} Y(x)dx\right)\right] =\exp\left[\sum_{n=1}^{\infty}\left(\frac{1}{n!}\int_0^{X_1}\cdots\int_0^{X_1} C[Y(x_1),\ldots,Y(x_n)] \times dx_1\cdots dx_n\right)\right]. \tag{B4}$$

For a Gaussian random process with mean 0, we have

$$\begin{aligned}&C[Y(x_1),Y(x_2)]=E[Y(x_1)Y(x_2)],\\ &C[Y(x_1),\ldots,Y(x_n)]=0, \quad \text{for } n=1 \text{ or } n>2.\end{aligned} \tag{B5}$$

Hence, for a Gaussian random process with mean 0, we have

$$E\left[\exp\left(\int_0^{X_1} Y(x)dx\right)\right] =\exp\left(\frac{1}{2}\int_0^{X_1} dX_2\int_0^{X_1} dX_3\, E[Y(X_2)Y(X_3)]\right). \tag{B6}$$

If $Q(x)$ is a square matrix of jointly Gaussian random processes with mean 0, we can write the following result by analogy with Eq. (B6);

$$E\left[S\left[\exp\left(\int_0^{X_1} Q(X_2)dX_2\right)\right]\right] =S\left[\exp\left[\frac{1}{2}\int_0^{X_1} dX_2\int_0^{X_1} dX_3\, E[Q(X_2)Q(X_3)]\right]\right], \tag{B7}$$

where S is the symmetrizing operator introduced in Sec. IV.

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